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A NOTE ON DIFFUSION FROM A LINE SOURCE IN A TURBULENT BOUNDARY LAYER

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NOMENCLATURE

- a, constant in the velocity profile power law;
- b, constant in the eddy diffusivity profile power law;
- C, concentration of diffusing matter;
- C_{max}, maximum concentration; concentration at the wall;
- G, rate of emission of diffusing matter per unit time and per unit length of line source;
- m, exponent in the velocity profile power law;
- n, exponent in the eddy diffusivity profile power law;
- u, velocity in the boundary layer;
- x, distance downstream from the source;
- y, distance normal to the wall.

Greek symbols

- ε, eddy diffusivity;
- λ, distance from the wall at which $(C/C_{max}) = 0.5$;
- ρ, density.

INTRODUCTION

THE RESULTS of an experimental investigation of diffusion of ammonia gas from a line source into a turbulent boundary layer was reported by Poreh and Cermak in an earlier paper [1]. Patankar and Taylor [2], in a short communication, compared those results with an analytical solution presented by Spalding [3] and found reasonable agreement in the intermediate zone where the plume is appreciably thinner than the veloicity boundary layer. However, the expression for the concentration profile did not match the experimental data very well. This was attributed to the fact that the velocity profile in [1] might have followed a $\frac{1}{6}$ or $\frac{1}{5}$ power law instead of the $\frac{1}{7}$ power quoted.

The purpose of this note is to show that the discrepancy is more likely due to the assumption in [3] of an eddy diffusivity which implies a constant shearing stress throughout the plume. Also it should be pointed out that an earlier solution, not mentioned by any of the above authors, was given by Sutton [4]. This solution gives the same results as quoted in [2] when the assumption of constant shearing stress is applied. Sutton's solution, however, is not limited to this case and if some other more reasonable assumptions about the eddy diffusivity are made the solution shows better agreement with the experimental data of [1] in both the intermediate and final zones.

SOLUTION OF THE DIFFUSION EQUATION

The steady state equation of diffusion in 2-dimensions, with the usual boundary layer assumptions, is

$$u\frac{\partial C}{\partial x} = \frac{\partial}{\partial y}\left(\varepsilon\frac{\partial C}{\partial y}\right). \tag{1}$$

For an infinite line source along $x = y = 0$, emitting at a constant rate G, the boundary conditions are

$$C \to 0 \text{ as } x, y \to \infty$$

$$C \to \infty \text{ along } x = y = 0$$

$$\varepsilon\frac{\partial C}{\partial y} \to 0 \quad y \to 0 \quad x > 0$$

$$\int_0^\infty \rho u c \, \mathrm{d}y = G \quad x > 0.$$

Sutton used power-law velocity and eddy diffusivity profiles $u = ay^m$ and $\varepsilon = by^n$ and the solution with a minor error corrected, is

$$C = \frac{G(m-n+2)}{a\Gamma\left(\dfrac{m+1}{m-n+2}\right)}\left(\frac{a}{(m-n+2)^2 bx}\right)^{\frac{m+1}{m-n+2}} \times \exp\left[-\frac{a}{b(m-n+2)^2}y^{m-n+2}\right]. \tag{2}$$

Therefore

$$\frac{C}{C_{\max}} = \exp[-0{\cdot}693(y/\lambda)^{m-n+2}]. \qquad (3)$$

COMPARISON WITH EXPERIMENT AND DISCUSSION

Assuming that the eddy diffusivity of mass varies with height in the same way as the eddy diffusivity of momentum and also that the shearing stress in the plume is constant, the so called Schmidt's conjugate power law applies, with $n = 1 - m$. Then.

$$C/C_{\max} = \exp[-0{\cdot}693(y/\lambda)^{2m+1}]. \qquad (4)$$

This is identical to equation (2.1) in [2] since m is equivalent to their $1/b$. The other relationships derived from equation (3) are

$$C_{\max} \sim x^{\frac{-m+1}{2m+1}}$$

and

$$\lambda \sim x^{\frac{1}{2m+1}}.$$

These are the same as (2.2) and (2.3) given by Patankar and Taylor.

Figure 2 in [2] shows that equation (4), with $m = \frac{1}{7}$ does not really fit the data of Poreh and Cermak but Patankar and Taylor stated that if the velocity profile followed a $\frac{1}{6}$ or $\frac{1}{5}$ power law, the agreement would be much better. However, it should be remembered that the region in which the shearing stress is roughly constant occupies only about one-tenth of the thickness of the turbulent boundary layer. Above this region, the shear stress decreases towards the edge of the boundary layer. From Figs. 5 and 7 of Poreh and Cermak's paper it can be seen that even in the intermediate zone the ammonia plume occupies from about $\frac{1}{4}$ to $\frac{4}{5}$ of the boundary layer. Obviously the assumption of constant shear stress throughout the plume is not valid.

Outside of the constant shear layer, the boundary layer behaves very much like a turbulent wake where the shear stress varies in such a way that the eddy diffusivity is more or less constant [5]. This indicates that in the final zone, the plume extends the entire thickness of the boundary layer and most of it lies in an area of constant eddy diffusivity. It is therefore reasonable to assume that in this zone the exponent n is equal to zero. Correspondingly, in the intermediate zone, one can assume a value of n between $(1 - m)$ and zero to give a shear stress decreasing with height.

Putting $m = \frac{1}{7}$ and $n = 0$ in equation (3) we get

$$C/C_{\max} = \exp[-0{\cdot}693(y/\lambda)^{\frac{15}{7}}]. \qquad (5)$$

This is plotted in Fig. 1 and agrees quite well with the experimental data of the final zone given in [1].

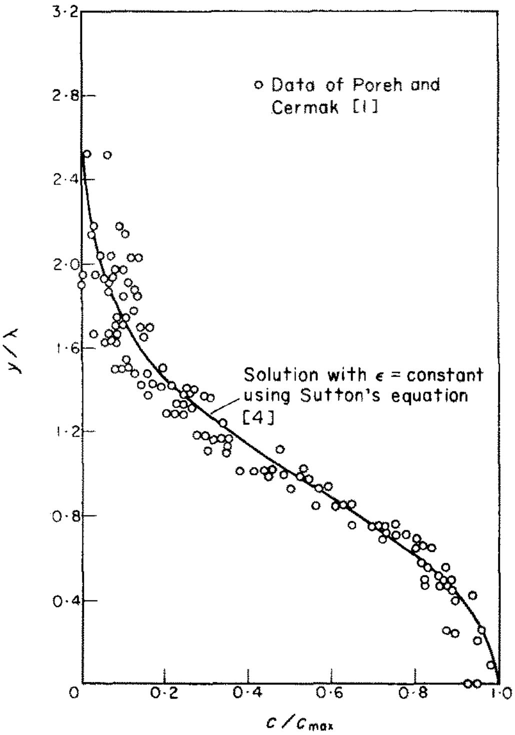

FIG. 1. Concentration profile in the final zone.

Measurements by Klebanoff [6] in a turbulent boundary layer showed the variation of shearing stress with distance from the wall. It can be seen that within the first $\frac{1}{4}$ of the boundary layer thickness, the shearing stress varies approximately as $(y/\delta)^{-\frac{1}{2}}$. Since the velocity distribution in [6] closely follows a $\frac{1}{6}$ power law, the value of n is approximately $\frac{1}{2}$. Taking a value of $n = \frac{4}{7}$ for the present case, equation (3) is plotted together with the experimental data of [1] in the intermediate zone and the equation given by Patankar and Taylor. It can be seen that the experimental data fits the curve with $n = \frac{4}{7}$ much better.

CONCLUSION

The comparisons made indicate that the discrepancy found in [2] is most likely due to the assumption of an eddy diffusivity which holds for only a small fraction of the plume near the wall. Using the assumption of an eddy diffusivity which is constant in the final zone and which gives a shearing stress decreasing with height in the intermediate zone the solution of Sutton can give a better description of the plume concentration.

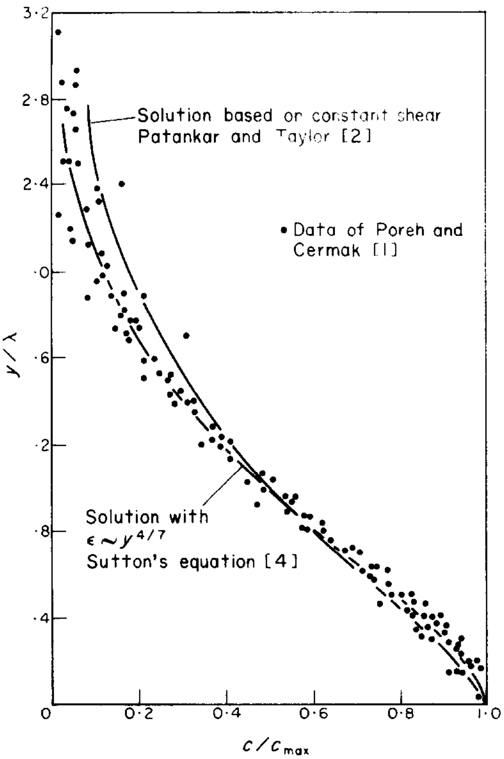

FIG. 2. Concentration profile in the intermediate zone.

It should be appreciated that the eddy diffusivity is actually varying in the streamwise direction as shown by Poreh and Cermak and the kind of solution given here does not represent the actual diffusion process exactly. Nevertheless, when applied judiciously, the simple solution given by Sutton can be very useful in practice.

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